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AMPLITUDE BEHAVIOR OF SHOCK WAVES IN A THERMOVISCOELASTIC SOLID

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Abstract—Recent theoretical studies of the propagation of one-dimensional shock waves in nonlinear viscoelastic materials have considered the influence of the strain gradient immediately behind a propagating shock front on the amplitude of the wave. In particular, the growth or decay behavior of the wave depends upon the relative magnitude of the strain gradient and a *critical strain gradient* which depends upon the material response properties. Here we apply the results of these studies to a particular constitutive model for a thermoviscoelastic material and determine the critical strain gradient and shock amplitude equation in terms of material response functions and thermophysical properties. The results are exhibited for a particular polymeric solid (PMMA) and compared to experimental results obtained in steady wave studies.

1. INTRODUCTION

IN A recent study of shock waves in nonlinear viscoelastic materials, Chen and Gurtin [1] considered the influence of thermodynamic effects on the growth and decay of a propagating shock front. They showed that the amplitude of a compressive shock front entering unstrained material at rest and at constant entropy depends upon a quantity λ called the *critical strain gradient* which itself depends on the mechanical and thermodynamic response functions of the material.

In the present study, we use a constitutive model developed for predicting steady-wave behavior in nonlinear thermoviscoelastic solids [2] along with response data for a particular polymeric solid (polymethyl methacrylate (PMMA)) to determine the critical strain gradient. The response data includes material functions obtained from experimental steady-wave studies, thermophysical properties and acoustic data. The calculated strain-dependent critical strain gradient is compared to experimental results obtained from steady shock wave experiments.

Further, assuming that the strain gradient behind the shock front remains constant, the amplitude variation of a propagating shock front is shown for the case where the wave is predicted to, (i) grow and (ii) decay.

2. SHOCK AMPLITUDE EQUATION—CRITICAL STRAIN GRADIENT [1]

In one dimension, a homogeneous viscoelastic material which does not conduct heat can be characterized by a constitutive assumption for the free energy ψ in terms of the

history of the strain $\varepsilon^t = \varepsilon(t-s)$ and the history of the absolute temperature $\theta^t = \theta(t-s) > 0$:

$$\psi = \hat{\psi}(\varepsilon^t, \theta^t).$$

Here, $\varepsilon = \partial_X u$, the function $u(X, t)$ being the displacement of the material point X at time t. Under suitable smoothness assumptions, it follows from the second law of thermodynamics that the stress σ and entropy η are given by (see Coleman [3])

$$\sigma = \hat{\sigma}(\varepsilon^t, \theta^t) = \partial_\varepsilon \hat{\psi}(\varepsilon^t, \theta^t), \tag{1}$$

$$\eta = \hat{\eta}(\varepsilon^t, \theta^t) = -\partial_\theta \hat{\psi}(\varepsilon^t, \theta^t), \tag{2}$$

where the partial differentiation is with respect to the present value of the argument. The internal energy is determined by

$$e = \hat{e}(\varepsilon^t, \theta^t) = \psi + \theta\eta. \tag{3}$$

Following Chen and Gurtin [1], we assume that for small relative strains γ and temperatures φ the material can be approximated by a linear viscoelastic material. Thus, if $(\varepsilon^t, \theta^t)$ are given histories, then

$$\hat{\sigma}(\varepsilon^t + \gamma^t, \theta^t + \varphi^t) = \hat{\sigma}(\varepsilon^t, \theta^t) + G(0)\gamma^t(0) + \int_0^\infty G'(s)\gamma^t(s)\,ds$$

$$+ F(0)\varphi^t(0) + \int_0^\infty F'(s)\varphi^t(s)\,ds + o(\|\gamma^t\| + \|\varphi^t\|) \tag{4}$$

as $(\|\gamma^t\| + \|\varphi^t\|) \to 0$. The quantities $G(s)$ and $F(s)$ are called the *stress–strain relaxation function* and the *stress–temperature relaxation function*, respectively. Further, we assume that $\hat{e}(\varepsilon^t + \gamma^t, \theta^t + \varphi^t)$ can be approximated by an analogous equation involving an *energy–strain relaxation function* $H(s)$ and an *energy–temperature relaxation function* $J(s)$. All of these functions depend, in general, on the underlying histories ε^t and θ^t.

Anticipating the appropriate underlying history for shock propagation, we define the jump history

$$(\varepsilon^t(s), \theta^t(s)) = (\varepsilon_I(s), \theta_I(s)) = \begin{cases} (\varepsilon, \theta), & s = 0, \\ (0, \theta_o), & s > 0 \end{cases} \tag{5}$$

with θ_o the reference temperature corresponding to the unstrained state. Then

$$\hat{\sigma}(\varepsilon_I(s), \theta_I(s)) = \sigma_I(\varepsilon, \theta),$$

$$\hat{e}(\varepsilon_I(s), \theta_I(s)) = e_I(\varepsilon, \theta)$$

are the *instantaneous stress* and *internal energy functions*. The quantities

$$(E_T)_I = \partial_\varepsilon \sigma_I = G(0), \tag{6}$$

$$A_I = \partial_\theta \sigma_I = F(0), \tag{7}$$

$$\varkappa_I = \partial_\theta e_I = J(0) \tag{8}$$

are called *instantaneous isothermal tangent modulus*, *instantaneous stress–temperature modulus*, and *instantaneous specific heat*, respectively. We assume, as is natural, that

$$(E_T)_I > 0, \qquad A_I \neq 0, \qquad \varkappa_I > 0. \tag{9}$$

In terms of $(E_T)_I$, the *isentropic tangent modulus* E_N is given by

$$E_N = (E_T)_I + \frac{\theta A_I^2}{\varkappa_I} > 0. \tag{10}$$

It will be useful in what follows to also define the *stress–energy modulus*†

$$\Gamma = \frac{A_I}{\varkappa_I}. \tag{11}$$

Here a shock wave is considered to be a propagating singular surface with velocity V across which the motion is continuous but the particle velocity $\dot{u}$, strain ε, temperature θ, and their derivatives suffer jump discontinuities. It is further assumed that the shock is compressive and that the wave is entering unstressed, unstrained material at constant and uniform temperature. Taking compressive stress and strain as positive, the following shock amplitude equation is obtained for the strain jump ε across the shock front:‡

$$\frac{d\varepsilon}{dt} = B\{\lambda - (\partial_X \varepsilon)\}, \tag{12}$$

where

$$B = \frac{2V\mu\xi}{3\xi + \mu(4-3\xi)}, \tag{13}$$

$$\xi = 1 - \frac{\rho_o V^2}{E_N}, \qquad \mu = \frac{2}{\varepsilon\Gamma} - 1, \tag{14}$$

and ρ_o is the reference density. The quantity λ, given by

$$\lambda = -\frac{1}{\xi V E_N}\{-G'(0)\varepsilon + F'(0)(\theta-\theta_o) - \Gamma[-H'(0)\varepsilon + J'(0)(\theta-\theta_o)]\}, \tag{15}$$

is called the *critical strain gradient*. We expect that for a reasonable range of strain levels $\lambda < 0$ (which is the case for PMMA, at least for $\varepsilon < 0{\cdot}04$). The functions B and λ are functions only of the strain amplitude since immediately behind a shock wave the temperature and the strain are related through the well-known Hugoniot relation

$$e_I(\varepsilon, \theta) - e_I(0, \theta_o) = \tfrac{1}{2}\sigma_I(\varepsilon, \theta)\varepsilon. \tag{16}$$

In order to establish the relative influence of λ and the strain gradient behind the shock $(\partial_X \varepsilon)$ on the wave amplitude ε, the sign of B must be determined. Using the Hugoniot relation (16), Nunziato and Herrmann [4] have shown that the shock velocity V is always subsonic with respect to the material behind the wave, i.e. $0 < \xi < 1$. If it is also assumed, as is the case for most materials, that $\Gamma > 0$ and that the Hugoniot stress–strain curve is one-to-one, then it can also be shown that $\mu > 0$.§ Thus, $B > 0$ and it follows that,‖ the *amplitude of a compressive shock wave entering unstrained material at rest and at uniform temperature will {grow, remain steady, decay} according to whether the critical strain gradient λ is {greater than, equal to, less than} the strain gradient behind the wave, $(\partial_X \varepsilon)$.*

† Γ is often called the Grüneisen ratio.
‡ Cf. Chen and Gurtin [1].
§ See, e.g. Nunziato and Herrmann [4]. Chen and Gurtin [1] follow a similar argument.
‖ Cf. Chen and Gurtin [1].

3. THERMOVISCOELASTIC MODEL

In a recent study of steady shock propagation in nonlinear viscoelastic solids [2], we specified a particular form for the constitutive equation for the free energy of the material which yielded the following constitutive equation for the stress σ at a point X in the material:

$$\sigma(t) = \sigma_E(\varepsilon, \theta) + \tfrac{1}{2}K'(\varepsilon)\{\mathscr{R}(\varepsilon^t) + \mathscr{J}(\theta^t)\}^2 + K(\varepsilon)\{\mathscr{R}(\varepsilon^t) + \mathscr{J}(\theta^t)\} \tag{17}$$

where

$$\mathscr{R}(\varepsilon^t) = \varepsilon + \int_0^\infty f'(s)\varepsilon(t-s)\,ds, \tag{18}$$

$$\mathscr{J}(\theta^t) = -\frac{1}{\theta_o}\left[g_o\theta + \int_0^\infty g'(s)\theta(t-s)\,ds\right]. \tag{19}$$

Similarly, the free energy yields an equation for the internal energy which takes the form

$$e(t) = e_E(\varepsilon, \theta) + \tfrac{1}{2}K(\varepsilon)\{\mathscr{R}(\varepsilon^t) + \mathscr{J}(\theta^t)\}^2 + K(\varepsilon)g_o\frac{\theta}{\theta_o}\{\mathscr{R}(\varepsilon^t) + \mathscr{J}(\theta^t)\}. \tag{20}$$

In (17) and (20), $\sigma_E(\varepsilon, \theta)$ and $e_E(\varepsilon, \theta)$ are *equilibrium* functions and represent the response of the stress and energy corresponding to constant values of strain and temperature for all time. The functions $K(\varepsilon)$, $f(s)$ and $g(s)$ are material response functions with $f(s) > 0$ a mechanical relaxation function and $g(s) > 0$ a thermal relaxation function. We assume $f(\infty) = g(\infty) = 0$ and write $g_o = g(0)$, $f_o = f(0)$, with $f(s)$ normalized such that $f_o = 1$.

To relate these response functions to the relaxation functions defined in the stress equation (4) and the analogous equation for the internal energy, it is necessary to compute the derivatives of (17) and (20) with respect to the relative strain/temperature history and evaluate them for the appropriate underlying history.† Carrying out this differentiation and evaluating the results for the jump history (5) yields the following identification between the relaxation functions;

$$G'(0) = -\frac{\theta_o\tau_\theta}{g_o\tau_\varepsilon}F'(0) = -\frac{1}{\tau_\varepsilon}[K'(\varepsilon)h(\varepsilon, \theta) + K(\varepsilon)],$$

$$H'(0) = -\frac{\theta_o\tau_\theta}{g_o\tau_\varepsilon}J'(0) = -\frac{K(\varepsilon)}{\tau_\varepsilon}(\varepsilon + g_o).$$

Here

$$h(\varepsilon, \theta) = \varepsilon - g_o\left(\frac{\theta}{\theta_o} - 1\right), \tag{21}$$

and we have assumed that the relaxation functions $f(s)$ and $g(s)$ can be represented as

$$f(s) = \exp\left(-\frac{s}{\tau_\varepsilon}\right), \qquad g(s) = g_o\exp\left(-\frac{s}{\tau_\theta}\right)$$

with τ_ε and τ_θ the characteristic mechanical and thermal relaxation times. Substituting these relations in (15) and noting that behind the shock $\theta = \Theta_I(\varepsilon)$, we have that

$$\lambda(\varepsilon) = -\frac{1}{\xi E_N V}\left\{\frac{g_o}{\tau_\theta}\left(\frac{\Theta_I}{\theta_o} - 1\right) + \frac{\varepsilon}{\tau_\varepsilon}\right\}\{K'(\varepsilon)h(\varepsilon, \Theta_I) + K(\varepsilon)[1 - \Gamma(\varepsilon + g_0)]\}. \tag{22}$$

† A similar procedure is carried out in detail by Schuler and Walsh [5] in a study regarding the critical strain gradient but using a mechanical model.

The moduli E_N and Γ can also be expressed in terms of the equilibrium functions and $K(\varepsilon)$. Evaluating the stress and energy relations, (17) and (20), for the jump history (5) we can compute the instantaneous moduli (6)–(8):

$$(E_T)_I = (E_T)_E + \tfrac{1}{2}K''(\varepsilon)h^2(\varepsilon,\theta) + 2K'(\varepsilon)h(\varepsilon,\theta) + K(\varepsilon),$$

$$A_I = A_E - \frac{g_o}{\theta_o}[K'(\varepsilon)h(\varepsilon,\theta) + K(\varepsilon)],$$

$$\varkappa_I = \varkappa_E - K(\varepsilon)g_o^2\frac{\theta}{\theta_o^2},$$

where

$$(E_T)_E = \partial_\varepsilon \sigma_E, \qquad A_E = \partial_\theta \sigma_E, \qquad \varkappa_E = \partial_\theta e_E$$

are the corresponding *equilibrium* moduli. Combining these relations with (10) and (11) and again noting that $\theta = \Theta_I(\varepsilon)$ yields the desired results.

Clearly then, to complete the formulation we need to evaluate the functions $\sigma_E(\varepsilon,\theta)$, $e_E(\varepsilon,\theta)$, $K(\varepsilon)$, $V(\varepsilon)$, $\Theta_I(\varepsilon)$, and the constants τ_ε, τ_θ and g_o. As it turns out, these quantities can be determined from data obtained by steady shock wave experiments, acoustic measurements and thermophysical properties measurements.

Material response functions

The plate-impact configuration has been used by Barker and Hollenbach [6] and by Schuler [7] to generate steady one-dimensional strain waves in a solid polymeric material. Using velocity interferometry, a time history of the particle velocity at a fixed location in the target sample is obtained for each impact level. Each history associated with a steady wave exhibits an instantaneous jump in particle velocity $\dot{u}_I$ followed by a smooth transition to an equilibrium value $\dot{u}_E$. Carrying out several tests over a range of impact levels and corresponding velocities V yields a set of measured values of V, $\dot{u}_I$ and $\dot{u}_E$, from which we can calculate, using a steady wave analysis [2, 7], the stress and strain at the head and the tail of a given wave. Thus, the stress at the head and the tail of the wave can be represented as a least-squares polynomial fit of the strain:

$$\begin{aligned} \sigma_I &= \bar{\sigma}_I(\varepsilon) = l_I\varepsilon + m_I\varepsilon^2 + n_I\varepsilon^3 + \ldots, \\ \sigma_E &= \bar{\sigma}_E(\varepsilon) = l_E\varepsilon + m_E\varepsilon^2 + n_E\varepsilon^3 + \ldots, \end{aligned} \tag{23}$$

with the coefficients corresponding to the equilibrium state $(0,\theta_o)$ ahead of the wave. The coefficients l_I and l_E are related to the instantaneous and equilibrium longitudinal sound speeds by†

$$l_I = \rho_o(C_I)_o^2, \qquad l_E = \rho_o(C_E)_o^2. \tag{24}$$

The function $\bar{\sigma}_I(\varepsilon)$ represents the Hugoniot stress–strain curve for the material and the shock velocity is determined by‡

$$\rho_o V^2(\varepsilon) = \frac{\bar{\sigma}_I(\varepsilon)}{\varepsilon}. \tag{25}$$

† Hereafter $(\,)_o$ is used to denote evaluation of a function at the equilibrium state $(0,\theta_o)$.

‡ Cf. Coleman, Gurtin and Herrera [8].

The internal energy corresponding to the head and the tail of a steady shock wave can be computed by

$$\begin{aligned} \bar{e}_I(\varepsilon) &= e_o + \tfrac{1}{2}\bar{\sigma}_I(\varepsilon)\varepsilon, \\ \bar{e}_E(\varepsilon) &= e_o + \tfrac{1}{2}\bar{\sigma}_E(\varepsilon)\varepsilon, \end{aligned} \tag{26}$$

where $e_I(0, \theta_o) = e_E(0, \theta_o) = e_o$. Equation $(26)_1$ is equivalent to (16) and, as we have already indicated, it implies that the temperature and the strain immediately behind the shock front are related. In view of the fact that at equilibrium, $\sigma = \sigma_E(\varepsilon, \theta)$, $e = e_E(\varepsilon, \theta)$, $(26)_2$ implies that the temperature and the strain are also related at the tail of a steady shock, i.e. $\theta = \Theta_E(\varepsilon)$. Here we assume that the instantaneous and equilibrium temperature–strain functions, $\Theta_I(\varepsilon)$ and $\Theta_E(\varepsilon)$, have the form

$$\begin{aligned} \Theta_I(\varepsilon) &= \theta_o\{1 + a_I\varepsilon + b_I\varepsilon^2\}, \\ \Theta_E(\varepsilon) &= \theta_o\{1 + a_E\varepsilon + b_E\varepsilon^2\}, \end{aligned} \tag{27}$$

where, as we have shown elsewhere [2], the coefficients can be evaluated from instantaneous energy deposition experiments involving a very low temperature rise [9] and from high-frequency acoustic wave experiments of the sound velocity as a function of temperature and pressure [10].

The material function $K(\varepsilon)$ can now be calculated using (20), evaluated for the jump history (5), and steady shock wave data:

$$K(\varepsilon) = \frac{2\{\bar{e}_I(\varepsilon) - e_E(\varepsilon, \Theta_I(\varepsilon))\}}{h(\varepsilon, \Theta_I(\varepsilon))\{h(\varepsilon, \Theta_I(\varepsilon)) + 2g_o[\Theta_I(\varepsilon)]/\theta_o\}}. \tag{28}$$

The form for the equilibrium response functions, $\sigma_E(\varepsilon, \theta)$ and $e_E(\varepsilon, \theta)$, is based upon an explicit assumption for the equilibrium internal energy function:

$$e_E(\varepsilon, \theta) = e_o + \int_{\theta_o}^{\theta} \varkappa_E(0, \omega)\, \mathrm{d}\omega + L(\varepsilon)G(\theta), \tag{29}$$

where $L(0) = 0$, $G(\theta_o) = 1$. Then, using the equilibrium counterparts of (1)–(3) along with (29), the equilibrium stress function is found to be given by

$$\sigma_E(\varepsilon, \theta) = M(\varepsilon)\frac{\theta}{\theta_o} - \theta L'(\varepsilon)\int_{\theta_o}^{\theta} \frac{G(\omega)}{\omega^2}\, \mathrm{d}\omega. \tag{30}$$

The strain-dependent functions $L(\varepsilon)$ and $M(\varepsilon)$ can be evaluated from the results of steady wave experiments, i.e. using (29) and (30) along with $(26)_2$ and $(23)_2$:

$$\begin{aligned} L(\varepsilon) &= \frac{1}{G(\Theta_E(\varepsilon))}\left\{\bar{e}_E(\varepsilon) - e_o - \int_{\theta_o}^{\Theta_E(\varepsilon)} \varkappa_E(0, \omega)\, \mathrm{d}\omega\right\}, \\ M(\varepsilon) &= \frac{\theta_o}{\Theta_E(\varepsilon)}\left\{\bar{\sigma}_E(\varepsilon) + \Theta_E(\varepsilon)L'(\varepsilon)\int_{\theta_o}^{\Theta_E(\varepsilon)} \frac{G(\omega)}{\omega^2}\, \mathrm{d}\omega\right\}. \end{aligned}$$

Using (30), it can be shown [2] that

$$G(\theta) = 1 + \int_{\theta_o}^{\theta} \frac{\omega}{\theta_o(A_E)_o}\partial_\theta A_E(0, \omega)\, \mathrm{d}\omega.$$

The remaining unknown functions $\varkappa_E(0, \theta)$ and $A_E(0, \theta)$ are evaluated from equilibrium thermophysical properties data.

Finally, the mechanical relaxation time τ_ε is obtained from acoustic data† while the thermal relaxation time τ_θ is determined by obtaining the best overall agreement of a steady wave analysis and experimentally observed steady wave profiles [2].

4. RESULTS FOR POLYMETHYL METHACRYLATE

As has been indicated in the above sections, and as one might expect, a not inconsiderable amount of material properties data is required to predict the shock amplitude behavior in a polymeric solid including thermal effects. In this section we show the results for the solid polymer, polymethyl methacrylate (PMMA). The strain dependence of the critical strain gradient is evaluated and compared to the strain gradient behind the wave measured directly from the experimental steady wave studies. Then, the amplitude variation of several shock fronts is calculated for the case where the magnitude of the strain gradient is (i) greater than and (ii) less than the critical strain gradient, assuming the strain gradient behind the wave remains constant as the wave front amplitude varies.

Throughout this section we take $\theta_o = 295°\text{K}$. For PMMA the results of plate-impact experiments by Barker and Hollenbach [6] and by Schuler [7] yield cubic fits for the Hugoniot stress–strain functions given by (23).‡ From (24), with $\rho_o = 1{\cdot}184\ \text{gm/cm}^3$, the sound speeds at the reference state are $(C_I)_o = 2{\cdot}763\ \text{mm}/\mu\text{sec}$, $(C_E)_o = 2{\cdot}731\ \text{mm}/\mu\text{sec}$.

The temperature dependence of the stress–temperature modulus $A_E(0, \theta)$ is found from the measured values of the equilibrium coefficient of linear thermal expansion β_E and the equilibrium isothermal bulk modulus B_E. Using data from Touloukian [12] for $\beta_E(\theta)$ and data from Heydemann and Guicking [13] for $B_E(\theta)$, $A_E(0, \theta)$ can be represented by§

$$A_E(0, \theta) = 3B_E(\theta)\beta_E(\theta) = A_0 + A_1\theta + A_2\theta^2 + A_3\theta^3.$$

The specific heat is determined using the relation

$$\varkappa_E(0, \theta) = \tilde{\varkappa}_E(\theta) - 3\theta A_E(0, \theta)\beta_E(\theta)$$

where $\tilde{\varkappa}_E(\theta)$, the specific heat at fixed pressure, is obtained from data given by Touloukian [12]; it can be represented by‖

$$\varkappa_E(0, \theta) = \varkappa_0 + \varkappa_1\theta + \varkappa_2\theta^2 + \varkappa_3\theta^3.$$

Using acoustic data obtained by Asay, Lamberson and Guenther [10], and electron beam data obtained by Perry [14], we can calculate $g_0 = 0{\cdot}191$ as well as the coefficients of the temperature–strain functions.¶ Also, from acoustic and steady wave studies, we have, for PMMA, $\tau_\varepsilon = 0{\cdot}22\ \mu\text{sec}$ and $\tau_\theta = 0{\cdot}10\ \mu\text{sec}$.

† The procedure for determining τ_ε from acoustic dispersion data is given by Nunziato and Sutherland [11].

‡ The values of the coefficients are: $l_I = 90{\cdot}43\ \text{kbar}$, $m_I = 681{\cdot}1\ \text{kbar}$, $n_I = -1396\ \text{kbar}$, $l_E = 88{\cdot}37\ \text{kbar}$, $m_E = 680{\cdot}7\ \text{kbar}$, $n_E = -2720\ \text{kbar}$.

§ The values of the coefficients are: $A_o = 0{\cdot}762 \times 10^{-2}\ \text{kbar/°K}$, $A_1 = 0{\cdot}163 \times 10^{-4}\ \text{kbar/°K}^2$, $A_2 = -1{\cdot}14 \times 10^{-8}\ \text{kbar/°K}^3$, $A_3 = -0{\cdot}727 \times 10^{-10}\ \text{kbar/°K}^4$.

‖ The values of the coefficients are: $\varkappa_o = -0{\cdot}460 \times 10^{-2}\ \text{kbar/°K}$, $\varkappa_1 = 1{\cdot}75 \times 10^{-4}\ \text{kbar/°K}^2$, $\varkappa_2 = -0{\cdot}688 \times 10^{-6}\ \text{kbar/°K}^3$, $\varkappa_3 = 0{\cdot}106 \times 10^{-8}\ \text{kbar/°K}^4$.

¶ The coefficients in equations (27) for PMMA are: $a_E = 0{\cdot}66$, $b_E = -1{\cdot}50$, $a_I = 0{\cdot}55$, $b_I = -0{\cdot}94$.

Figure 1 shows the shock amplitude dependence ε of several of the *instantaneous* material functions; the isentropic tangent modulus E_N, the shock velocity V, the temperature Θ_I, and the stress–energy modulus Γ.

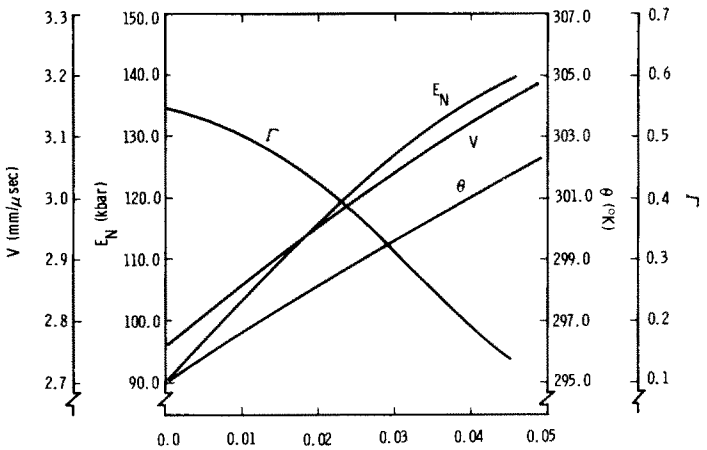

FIG. 1. Shock velocity, isentropic tangent modulus, temperature, and stress–energy modulus as functions of strain amplitude for PMMA.

Critical strain gradient

We wish to compare the values of the critical strain gradient $\lambda(\varepsilon)$, calculated by (22), with experimental studies, in particular, the steady shock wave studies using PMMA. Since the waves were steady, the value of the strain gradient behind the wave should be the critical strain gradient. Figure 2 shows the comparison. The solid line indicates the strain-dependence of λ using the constitutive model, results of steady wave experiments, and thermophysical data, as described above. Several steady wave experiments were made at both the 0·02 and 0·04 levels of initial strain. The circles at these strain levels represent the mean values of the strain gradients obtained graphically from the experimental data.† The error brackets reflect not only the difference between the results for several tests at the same nominal impact level, but also the difference in the graphical determination of the slopes by three persons. There was only one steady wave experiment performed at the 0·03 level of initial strain and there was no variation in value of the critical strain gradient determined. The correlation between the mean values of the experimental studies and the calculated values is good.

The present values of λ are somewhat higher than the results for the purely mechanical model obtained by Schuler and Walsh [9] and indicate the influence of the thermodynamic effects on shock propagation in nonlinear polymeric materials.

In the limit of zero strain amplitude at the shock, the value of the critical strain gradient has a magnitude twice that of the critical amplitude of an acceleration wave [1]. That value determined from our results predicts a critical distance at which an acceleration

† Actually, since the experimental measurements represent time histories of particle velocity, the graphical determination of the slope behind the wave front represents a critical acceleration λ_c which is related to λ by $\lambda = \lambda_c/V^2$.

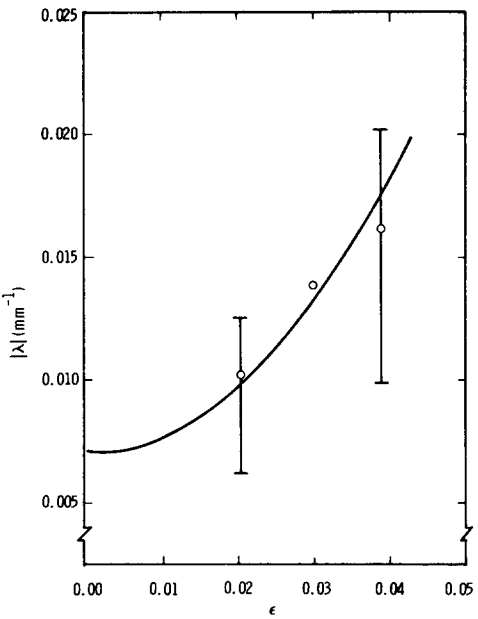

FIG. 2. Comparison of calculated and experimentally determined critical strain gradient for PMMA (solid line represents calculated values).

wave will (appear to) form a shock discontinuity in PMMA which is consistent with experimental observations (see Walsh and Schuler [15]).

Shock amplitude variation

Figure 3 shows the amplitude variation of a shock front of initial strain amplitude $\varepsilon = 0{\cdot}03$ for several values of strain gradient behind the wave. For $\varepsilon = 0{\cdot}03$ the value of the critical strain gradient is $\lambda = -0{\cdot}0132$. Thus, assuming a value of strain gradient

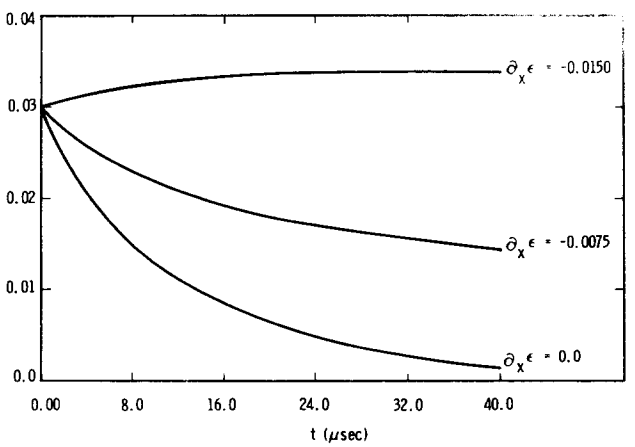

FIG. 3. Strain amplitude at the shock as a function of time for several values of strain gradient behind the wave; $\varepsilon(0) = 0{\cdot}03$.

$\partial_X \varepsilon = -0.015$ should result in the wave amplitude increasing, which is the case. Of course, as shown in Fig. 2, $|\lambda|$ increases as ε increases, thus the rate of increase of amplitude decreases, with the amplitude approaching the value $\varepsilon^* = 0.034$ for which $\lambda(\varepsilon^*) = -0.015$. Similarly, $\partial_X \varepsilon = -0.0075$ and $\partial_X \varepsilon = 0$ result in a wave front of decreasing amplitude. Clearly, $\partial_X \varepsilon = \lambda$ results in a steady wave front with amplitude $\varepsilon = 0.03$. Although, in general, the strain gradient behind the wave does not remain constant, this procedure does give some indication of the type of behavior which might be expected (see Ref. [16]).

5. CONCLUSIONS

The results of the theoretical study of Chen and Gurtin indicate the existence of a critical strain gradient for shock propagation in a general class of nonlinear viscoelastic materials including thermodynamic influences. Here we use these results to determine the critical strain gradient for a particular viscoelastic solid. A specific constitutive form was used which has given favorable results in predicting steady wave behavior for shock waves in the solid polymer, PMMA. The critical strain gradient calculated compares favorably with that observed from steady shock wave experiments. As mentioned earlier, the consideration of thermodynamic influences yields values for the critical strain gradient somewhat higher than those determined neglecting thermal effects.

We feel the correlation evident in these results provides some further verification with regard to the determination of the material response functions and the results give further insight into the behavior of shock waves in nonlinear viscoelastic solids.

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Абстракт—Последные теоретические исследования касающиеся распространения одноразмерных ударных волн в нелинейных вязкоупругих материалах обсуждают влияние градиента деформации непосредственно после распространяющегося фронта удара на амплитуду волны. В особенности, поведение роста или затухания волны зависит от относительной величины градиента деформации и *критического градиента деформации*, который находится в зависимости от свойств реагирования материала. В работе применяются результаты этих исследований для частной определяюшей модели из термовязкоупругого материала. Определяются критический градиент деформации и уравнение амплитуды удара, в виде функций характеристик материала и термофизических свойств. Показываются результаты для частного полимерного твердого тела и сравниваются с экспериментальными результатами для стационарной волны.